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On cross-waves

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Cross-waves are standing waves with crests at right angles to a wave-maker. They generally have half the frequency of the wave-maker and reach a steady state at some finite amplitude. A second-order theory of the modes of oscillation of water in a tank with a free surface and wave-makers at each end leads to a form of Mathieu's equation for the amplitude of the cross-waves, which are thus an example of parametric resonance and may be excited at half the wave-maker frequency if this is within a narrow band. The excitation depends on the amplitude of the wave-maker at the surface and the integral over depth of its amplitude. Cross-waves may be excited even if the mean free surface is stationary. The effects of finite amplitude are that the cross-waves approach a steady state such that a given amplitude is achieved at a frequency greater than that for free waves by an amount proportional to the amplitude of the wave-maker. The theory agrees reasonably well with the experimental results of Lin & Howard (1960). The amplification of the cross-waves may be understood in terms of the rate of working of the wave-maker against transverse stresses associated with the cross-waves, one located at the surface and the other equal to Miche's (1944) depth-independent second-order pressure. The theory applies to the situation where the primary motion consists of standing waves and the cross-waves are constant in amplitude away from the wave-maker, but certain generalizations may be made to the situation where the primary waves are progressive and the cross-waves decay away from the wave-maker.

1. Introduction

Faraday (1831*a*) observed that if the edge of a vibrating wooden plate† was immersed about $\frac{1}{8}$ in. into a basin of water, then "Elevations, waves or crispations immediately formed but of a peculiar character. Those passing from the surface of the plate over the water to the sides of the basin were hardly sensible, but apparently permanent elevations formed, beginning at the plate and projecting directly out from it to the extent of $\frac{1}{3}$ or $\frac{1}{2}$ an inch or more, like the teeth of a very short coarse comb" (entry 118 for 1 July 1831). In an elaboration of this experiment at a lower frequency, Faraday noted that the waves had a frequency half that of the vibration of the plate (entry 140 for 5 July 1831; see also Faraday 1831*b*).

† Faraday caused the vibrations by stroking a wet glass rod, one end of which was held against the plate. The experiment is readily reproduced.

These waves, with crests at right angles to the wave-maker and generally half the frequency, have come to be known as 'cross-waves', and on a larger scale appear to be a common phenomenon in wave tanks in which the width of the wave-maker is rather larger than the wavelength of the primary waves being generated. Schuler (1933) and Spens (1956) studied cross-waves in wave tanks with beaches at the end opposite the wave-maker, in which case the primary waves were progressive and the amplitude of the cross-waves decayed away from the wave-maker. Lin & Howard (1960) carried out a thorough experimental investigation of the cross-waves generated in a tank with a rigid wall opposite the wave-maker, in which case the primary motion consisted of standing waves and the cross-waves were constant in amplitude away from the wave-maker. Lin & Howard were unable to account for the excitation of the cross-waves theoretically, though they did develop a theory which seemed to explain why the cross-waves should have half the frequency of the wave-maker.

This paper is directed mainly towards an understanding of the experimental results of Lin & Howard (1960), but the solution of this problem leads to at least a partial theory of the cross-waves generated in the case in which the primary waves are progressive, and to a suggestion as to how they might be suppressed.

One must answer the following questions:

(i) How can waves with crests at right angles to a wave-maker be generated?

(ii) Why should such waves have half the frequency of the wave-maker?

(iii) What determines the amplitude at which cross-waves reach a steady state?

The half-frequency property of cross-waves suggests that they may be an example of parametric resonance. This well-known phenomenon (see e.g. Bogoliubov & Mitropolsky 1961) refers to the excitation of an oscillator the governing parameters of which are varied with a frequency sufficiently close to $2\omega/N$, where ω is the frequency of free oscillation and N is integral. The strongest resonance is for $N = 1$. Mathematically the phenomenon is described in its simplest form by Mathieu's equation. A simple mechanical example of parametric resonance is the excitation of a simple pendulum by periodic vertical motion of its point of support. This may be understood physically in terms of the work done against the centrifugal force exerted by the pendulum.

An example of parametric resonance in fluid mechanics was described by Benjamin & Ursell (1954) who showed that in a vertically oscillating container of water the amplitude of a standing surface wave satisfies Mathieu's equation. It is not difficult to show that in this case the excitation of the surface wave may be interpreted in terms of the work done against the pressure exerted on the bottom of the container by the surface wave.

In this paper, I start with roughly the same formulation of the problem as Lin & Howard (1960) and show that if quadratic terms are retained in the free-surface conditions a function related to the amplitude of cross-waves satisfies Mathieu's equation. Mathematically this accounts for the excitation of half-frequency cross-waves at certain frequencies of the wave-maker; the excitation may be understood physically in terms of the work done by the wave-maker against transverse stresses associated with the cross-waves. It is this interpretation which suggests how cross-waves may be generated when the primary waves are progressive. It will be shown how cross-waves approach a steady state at finite amplitude, as is usual for a parametrically resonant non-linear oscillator; the dependence of this amplitude on frequency permits a comparison of the theory with the experimental data of Lin & Howard (1960).

2. Formulation

A rectangular tank of water has wave-makers at $x^* = 0, 2L$, rigid sides at $y^* = 0, W$, a rigid bottom at $z^* = -H$ and a free surface with undisturbed position $z^* = 0$. The experimental tank used by Lin & Howard corresponds to one half of this (i.e. due to the symmetry of the problem they had a rigid wall at $x^* = L$). The asterisk denotes dimensional variables.

The flow is assumed to be incompressible, inviscid and irrotational, with a velocity potential ϕ^*. We non-dimensionalize with respect to a length L/π and time $(L/g\pi)^{\frac{1}{2}}$ as follows:

$$\left.\begin{aligned} x = \frac{\pi x^*}{L}, \quad y = \frac{\pi y^*}{L}, \quad z = \frac{\pi z^*}{L}, \\ t = \left(\frac{\pi g}{L}\right)^{\frac{1}{2}} t^*, \quad \phi = \left(\frac{\pi}{L}\right)^2 \left(\frac{L}{\pi g}\right)^{\frac{1}{2}} \phi^*, \end{aligned}\right\} \tag{2.1}$$

and define
$$l = \frac{L}{W}, \quad h = \frac{\pi H}{L}. \tag{2.2}$$

The velocity $\mathbf{u} = \nabla\phi$, where
$$\nabla^2\phi = 0. \tag{2.3}$$

The conditions on the free surface $z = \zeta(x,y,t)$ are

$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y, \tag{2.4}$$

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + \zeta = 0. \tag{2.5}$$

The normal velocity vanishes on the fixed sides and bottom of the tank. Thus

$$\phi_y = 0 \quad \text{on} \quad y = 0, \quad \pi/l, \tag{2.6}$$

$$\phi_z = 0 \quad \text{on} \quad z = -h. \tag{2.7}$$

The normal velocity is prescribed on the wave-makers, which are displaced to $x = F(z)\beta(t)$ and $x = 2L - F(z)\beta(t)$. For small amplitude of the wave-maker motion, we may linearize this and apply the boundary conditions

$$\phi_x = F(z)\dot{\beta} \quad \text{on} \quad x = 0, \quad \phi_x = -F(z)\dot{\beta} \quad \text{on} \quad x = 2\pi. \tag{2.8}$$

The dot superscript describes differentiation with respect to t of a function of t alone.

3. The cross-wave equations

To second order in the amplitude of ϕ, ζ, the free surface conditions may be taken as

$$\phi_z + \zeta\phi_{zz} = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y, \quad \text{on} \quad z = 0, \tag{3.1}$$

$$\phi_t + \zeta\phi_{zt} + \tfrac{1}{2}|\nabla\phi|^2 + \zeta = 0, \quad \text{on} \quad z = 0. \tag{3.2}$$

We now describe the motion in the tank as the sum of an $O(\alpha)$ primary motion, independent of y and satisfying the inhomogeneous boundary conditions (2.8), plus all the free modes of the tank with some dependence on y, i.e. we write

$$\zeta = \zeta_0(x,t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{m,n}(t)\cos mx \cos nly, \tag{3.3}$$

$$\phi = \phi_0(x,t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} d_{m,n}(t)\cos mx \cos nly \exp\{(m^2+n^2l^2)^{\frac{1}{2}} z\}. \tag{3.4}$$

We shall concentrate on the fundamental cross-wave ($m = 0$, $n = 1$), assuming $c_{0,1}$ and $d_{0,1}$ to be $O(\epsilon)$ and all the other coefficients to be second order in ϵ and α. We then substitute (3.3), (3.4) into (3.1), (3.2), average in the x direction, and consider the coefficients of $\cos ly$ to second order. Hence

$$ld(1+al)+bc = \dot{c}, \tag{3.5}$$

$$\dot{d}(1+al)+\ddot{a}c+\dot{a}ld+c = 0, \tag{3.6}$$

where $c = c_{0,1}$, $d = d_{0,1}$ and

$$a = \overline{\zeta_0}|_{z=0}, \quad b = \overline{\phi_{0zz}}|_{z=0}, \quad \overline{\phi_{0zt}}|_{z=0} = \ddot{a}, \tag{3.7}$$

the bar denoting the average over x. But clearly the mean free surface displacement is given by the amount of water displaced by the wave-makers, so that

$$a = \frac{1}{\pi}\int_{-\infty}^{0} F(z)\,dz\beta. \tag{3.8}$$

Similarly, the mean vertical velocity at any level is given by

$$\overline{\phi_{0z}}(z) = \frac{1}{\pi}\int_{-\infty}^{z} F(z)\,dz\dot{\beta}, \tag{3.9}$$

so that

$$b = (1/\pi)F(0)\beta. \tag{3.10}$$

Eliminating d from (3.5, 6) we have

$$\ddot{c} - b\dot{c} + l(1+\ddot{a}-\dot{b}/l)c = 0. \tag{3.11}$$

In the absence of the wave-makers ($a = b = 0$), this equation describes free oscillations of frequency $l^{\frac{1}{2}}$. In the presence of wave-makers the implications of (3.11) are best understood by defining

$$f(t) = c\exp\left[-\int^{t} b(\tau)\,d\tau\right] = c\exp\left[-\tfrac{1}{2}\beta F(0)/\pi\right], \tag{3.12}$$

whence, retaining only linear and quadratic terms,

$$\ddot{f}+l(1+\ddot{a}-\tfrac{1}{2}\dot{b}/l)f = 0, \tag{3.13}$$

or

$$\ddot{f}+l\left\{1+\frac{\ddot{\beta}}{2l\pi}\left[2l\int_{-\infty}^{0} F(z)\,dz - F(0)\right]\right\}f = 0. \tag{3.14}$$

If $\beta = \alpha\cos\sigma t$ this becomes Mathieu's equation,

$$\ddot{f}+l(1+\gamma\cos\sigma t)f = 0, \tag{3.15}$$

where

$$\gamma = \frac{\alpha\sigma^2}{2l\pi}\left[F(0) - 2l\int_{-\infty}^{0} F(z)\,dz\right]. \tag{3.16}$$

Half-frequency cross-waves are parametrically excited for σ in the range

$$l^{\frac{1}{2}}\left\{2 \pm \frac{\alpha}{\pi}\left[F(0) - 2l\int_{-\infty}^{0} F(z)\,dz\right]\right\}, \tag{3.17}$$

and the growth rate is $O(\alpha)$ within this band.

The degree of excitation depends on the amplitude of the wave-maker at $z = 0$, and the integral of the wave-maker amplitude over the depth of the water. It is somewhat surprising that this integral is not weighted with the energy density of the cross-waves, this will be discussed further in §6. We note three special cases:

(i) $F(0) = 0$, i.e. the wave-maker is totally submerged. The excitation then depends only on the acceleration of the mean free surface, and the problem resembles that of Benjamin & Ursell, though in their case the whole tank was accelerating.

(ii) $\int_{-\infty}^{0} F(z)\,dz = 0$, an example of such a wave-maker would be a flap pivoting about an axis midway between the free surface and the lower edge of the flap. In this case, cross-waves are still excited even though the acceleration of the mean free surface is zero. This is an important example as it demonstrates the essential difference between cross-waves and the surface standing waves examined by Benjamin & Ursell.

(iii) $F(0) - 2l\int_{\infty}^{0} F(z)\,dz = 0$, so that cross-waves are suppressed (at least to this order).

Although Mathieu's equation has growing solutions for σ close to $2l^{\frac{1}{2}}/N$, where N is an integer other than 1, (3.15) cannot be used to investigate such a resonance for cross-waves. The reason is that the growth rate at such a resonance is $O(\alpha^N)$, which might just as easily be produced by neglected terms of $O(\alpha^M\epsilon)$ (provided N/M is integral) as by the quadratic terms which have been retained. For example, for $N = 2$ the full equation for the amplitude of the cross-wave would contain terms with coefficients $O(\alpha^2)$ and frequency 2σ. These could give a growth rate $O(\alpha^2)$, comparable with that from the coefficient of $O(\alpha)$ and frequency σ. Cross-waves can probably be excited for N an integer other than 1, but a quantitative study of this would require analysis of the problem to order N in α.

So far only the fundamental cross-wave ($n = 1$) has been discussed. Higher modes may be excited in just the same way, the governing equations would be just as before with l replaced by nl. More complicated modes corresponding to $c_{m,n}$ and $d_{m,n}$ in (3.3), (3.4) with neither m nor n zero are probably also subject to parametric resonance, but away from the frequency bands in which this can occur $c_{m,n}$, $d_{m,n}$ merely describe small disturbances forced by the primary motion and whatever cross-wave, if any, is being excited, and will thus be small compared with the coefficients describing the cross-wave, as assumed earlier.

Lin & Howard (1960) derived an equation equivalent to (3.11) for the particular case $F(z) = e^{\frac{1}{2}z}$ and $\beta = \alpha\cos\sigma t$, but they then looked for steady solutions and found that these were only possible with frequency $\frac{1}{2}\sigma$ and if a relation between σ^2 and l was satisfied. They could not account for the excitation of cross-waves, though we see now that their approach is equivalent to demanding that σ should be at one end of the frequency band within which half-frequency parametric resonance occurs.

The analysis leading to (3.17) is easily modified for finite depth h, though the algebra is more tedious. The end result is that half-frequency cross-waves are excited for σ in the range,

$$(l \tanh lh)^{\frac{1}{2}} \left\{ 2 \pm \frac{\alpha}{\pi} \left[F(0) - \frac{7 \tanh^2 lh - 3}{2 \tanh lh} \, l \int_{-h}^{0} F(z)\, dz \right] \right\}. \tag{3.18}$$

4. Finite-amplitude effects

If we were to take into account third-order terms in the free surface conditions and terms arising from a proper expansion of the boundary conditions on the wave-makers (rather than the linearized (2.8)), then the equation governing the amplitude of the fundamental cross-wave would contain extra terms of $O(\epsilon^3)$ and $O(\epsilon\alpha^2)$ (nothing of $O(\epsilon^2\alpha)$ or $O(\alpha^3)$ would appear). The term of $O(\epsilon\alpha^2)$ would

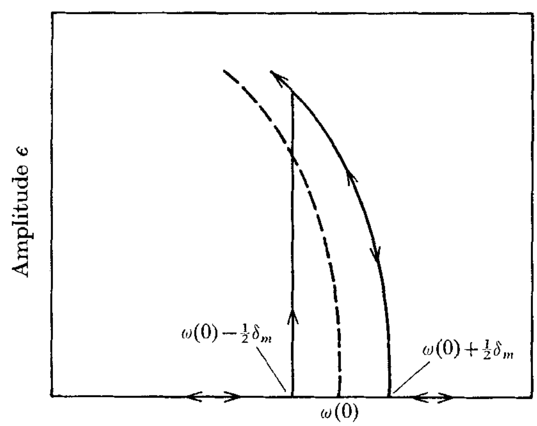

FIGURE 1. Frequency-amplitude curve of a non-dissipative parametrically resonant non-linear oscillator. Arrows indicate the direction in which the frequency is changing.

alter the width of the parametric resonance by a fraction of $O(\alpha)$, which we neglect, and also alter the basic frequency of the cross-wave by $O(\alpha^2)$. The term of $O(\epsilon^3)$ would also alter the frequency of the cross-wave by $O(\epsilon^2)$, indeed this is just saying that the frequency of a free standing wave in a tank is amplitude-dependent. This is crucial, for it means that instead of growing indefinitely, if σ is within the frequency band for resonance, the cross-wave will approach a steady state at some finite amplitude. The theory of a parametrically resonant non-linear oscillator is well known (e.g. Bogoliubov & Mitropolsky 1961, §17), but the relevant points will be summarized briefly here.

Suppose that the oscillator has a natural frequency $\omega(\epsilon)$ which is a decreasing function of its amplitude ϵ (as for cross-waves), as shown by the broken line in figure 1. Suppose further that half-frequency parametric resonance occurs for σ

within $\pm\delta_m$ of twice the frequency of the free oscillator. If σ is now decreased slowly from some value greater than $2\omega(0)+\delta_m$ then the instability of the oscillator will first appear, with frequency $\frac{1}{2}\sigma$, when $\sigma = 2\omega(0)+\delta_m$. If we continue to decrease σ by a small amount at a time, at each stage the oscillator will have frequency $\frac{1}{2}\sigma$ and will grow in amplitude until $\omega(\epsilon) = \frac{1}{2}\sigma - \frac{1}{2}\delta_m$. Thus the frequency-amplitude curve of the oscillator will lie a distance $\frac{1}{2}\delta_m$ to the right of the curve for free oscillations. If σ increases slowly from a value less than $2\omega(0)-\delta_m$, the oscillator will not respond until $\frac{1}{2}\sigma$ reaches $\omega(0)-\frac{1}{2}\delta_m$, at which frequency the frequency-amplitude curve will jump to that for σ decreasing. This response of the oscillator to parametric resonance is sketched in figure 1. In practice dissipative and higher-order non-linear effects generally affect the oscillator for large ϵ (see Bogoliubov & Mitropolsky for the effect that this has on figure 1).

5. Comparison with experiment

The results of §§3, 4 enable us to give a precise description of the implications of the theory for the experiments of Lin & Howard (1960). In these experiments they used just one wave-maker and a rigid wall at $x = \pi$, but by symmetry the present theory applies. The wave-maker extended to the bottom of the tank and was of the flap type, i.e. $F(z) = -z/h$ (so that α is the amplitude of the wave-maker at $z = 0$). The tank was sufficiently deep for $\tanh lh$ to be very close to 1. Thus the parametric excitation of the fundamental cross-wave is described by

$$\ddot{f} + l\left[1 + \frac{2\alpha}{\pi}(1-lh)\cos\sigma t\right]f = 0, \tag{5.1}$$

and half-frequency cross-waves are amplified for

$$l^{\frac{1}{2}}\left(2 - \frac{\alpha}{\pi}|1-lh|\right) < \sigma < l^{\frac{1}{2}}\left(2 + \frac{\alpha}{\pi}|1-lh|\right). \tag{5.2}$$

The wave growth would be limited by finite-amplitude effects, as discussed in §4. If the cross-waves are investigated by slowly decreasing the wave-maker frequency (as seems to have been the case in the experiments of Lin & Howard), allowing time for a steady state to be reached at each stage and then measuring the amplitude and frequency of the cross-waves (if present), then their frequency should be half that of the wave-maker (as was observed), and the frequency-amplitude curve should be to the right of that for free oscillations by a frequency $\frac{1}{2}\delta_m$ given by

$$\tfrac{1}{2}\delta_m = \frac{\alpha l^{\frac{1}{2}}}{2\pi}|1-lh|. \tag{5.3}$$

Qualitatively this is in agreement with the experimental results of Lin & Howard. They obtained frequency-amplitude data which were generally to the right of the curves for free oscillations (obtainable from the theory of Penney & Price 1952) by an amount which increased as the amplitude of the wave-maker increased.

In Lin & Howard's experiment $W = 24\frac{3}{16}$ in., $H = 24$ in., so that

$$\tfrac{1}{2}\delta_m/l^{\frac{1}{2}} = 0{\cdot}341\alpha. \tag{5.4}$$

$\frac{1}{2}\delta_m/l^{\frac{1}{2}}$, i.e. the frequency shift for a given cross-wave amplitude, may be measured approximately from the data of Lin & Howard and plotted against α. In figure 2 the vertical lines join the greatest and least values of $\frac{1}{2}\delta_m/l^{\frac{1}{2}}$ for a given value of α.

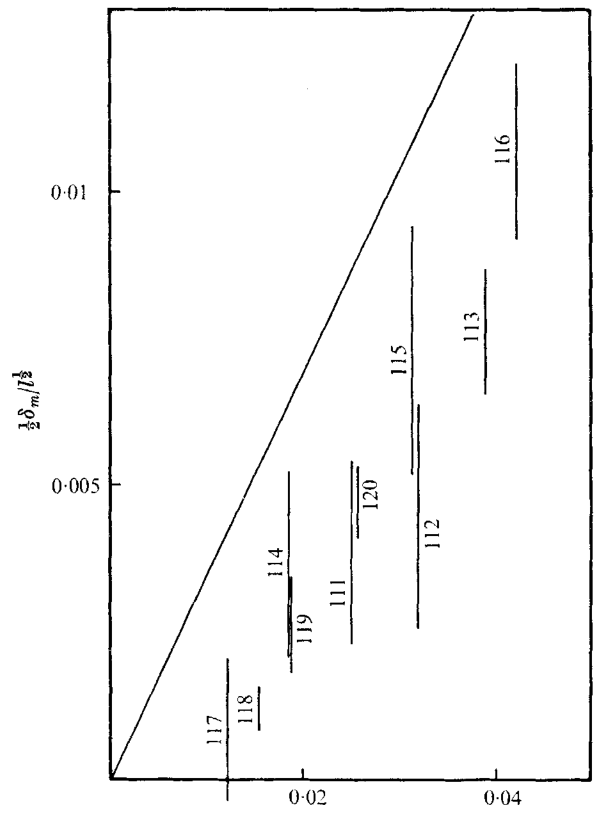

FIGURE 2. Frequency shift of frequency-amplitude curves of Lin & Howard (1960) *versus* wave-maker amplitude α. The straight line through the origin represents (5.4), the numbered vertical lines give the spread of the data for a run of that number. Runs 111, 112, 113 were for $l = 0{\cdot}289$, runs 114, 115, 116 for $l = 0{\cdot}362$ and runs 117, 118, 119, 120 for $l = 0{\cdot}560$.

No values of $\frac{1}{2}\delta_m/l^{\frac{1}{2}}$ have been measured for runs in which $4l$ was integral, as in these cases the data were more scattered and the frequency amplitude curves were not the same in shape as the curves for free oscillation. This was probably because the primary standing wave was then very large due to resonance, and third-order terms in the cross-wave equation were important, even though formally $O(\epsilon\alpha^2)$.† The straight line in figure 2 represents (5.4).

We see that for each value of l the slope of the data is consistent with (5.4), but the origin is not. Thus the data suggests that the theory adequately describes

† It is clear that a steady state is reached at $\epsilon = O(\alpha^{\frac{1}{2}})$, so that the cross-waves are generally larger than the $O(\alpha)$ primary waves, and third-order terms $O(\epsilon^3)$ are larger than $O(\epsilon\alpha^2)$ terms.

the strength of the parametric resonance, but that the frequency of free cross-waves was overestimated by an amount independent of α, and less than $\frac{1}{2}\%$.

Viscous effects would cause a decrease in δ_m, though in a tank of the size used by Lin & Howard this should be negligible.

Further evidence for our interpretation of the cross-wave as a parametrically excited non-linear oscillator comes from the phase relation between the wave-maker and cross-wave. Here $1-lh < 0$, and when a steady cross-wave is set up it is such that σ is at the upper end of the frequency band in which resonance occurs. Thus if the wave-maker displacement inwards is proportional to $\cos \sigma t$, f, and hence c, is proportional to $\sin \frac{1}{2}\sigma t$ (see the appendix). Thus, when the wave-maker is fully in, the surface displacement due to the cross-wave is zero, and, when the wave-maker is out, the free surface displacement due to the cross-wave is at a maximum or minimum. This agrees with the observations of Lin & Howard. Indeed, the accuracy with which it is true emphasises the unimportance of viscosity. For in the presence of slight damping like $e^{-\mu t}$ the cross-wave response at the end of the resonance band would be $\sin(\frac{1}{2}\sigma t+\theta)$ where $\theta = \mu/\delta_m$ (see the appendix). θ was too small to measure in Lin & Howard's experiments, so that dissipative effects were indeed negligible.

6. Energy considerations

In §3 we saw that the cross-waves are coupled to the primary motion through the non-linear free surface conditions and actually depend on the spatial mean of the position of the free surface and the vertical strain there. However, this does not really tell us how energy is being fed into the cross-waves. Clearly the energy of the cross-waves must be derived either from the primary motion or directly from the wave-makers. I shall now show that the latter is the case and explain how it occurs.

Before the cross-waves are set up the primary motion is $O(\alpha)$ with energy $O(\alpha^2)$. It is clear from the analysis of §3 that when cross waves of $O(\epsilon)$ and energy $O(\epsilon^2)$ are established, the primary motion is affected only to second order, thus the change in its energy is third order, insufficient to supply the energy of the cross-waves. Thus the cross-wave energy must be derived directly from the wave-makers.

An energy equation for the cross-waves may be derived by multiplying (3.11) by $(\pi^2/l^2)\dot{c}$. After some manipulation we have

$$\frac{d}{dt}\left\{\frac{1}{2}\frac{\pi^2}{l^2}\dot{c}^2+\frac{1}{2}\frac{\pi^2}{l}c^2-\frac{\pi}{l^2}\left[F(0)-l\int_{-\infty}^{0}F(z)\,dz\right]\dot{\beta}c\dot{c}\right\}$$
$$= -\frac{\pi}{l^2}F(0)\dot{\beta}c\ddot{c}+\frac{\pi}{l}\int_{-\infty}^{0}F(z)\,dz\dot{\beta}(\dot{c}^2+c\ddot{c}). \qquad (6.1)$$

The left-hand side of this equation describes the rate of change of the cross-wave energy within the volume $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant \pi/l$, $-\infty < z$. The energy is made up of the kinetic and potential energy of the cross-wave plus a third-order (and hence negligible) interaction term involving the velocity of the wave-maker. The right-hand side may be interpreted as the rate of working of each wave-maker against a stress $-\frac{1}{2}\pi/l^2 c\ddot{c}$ at the surface, and a stress $\frac{1}{2}(\pi/l)(\dot{c}^2+c\ddot{c})$ uniformly distributed with depth.

At first sight this is surprising. One expects the wave-makers to be working against the second-order pressure associated with the cross-waves, but this pressure involves a term proportional to e^{2lz} so that a term $\int_{-\infty}^{0} F(z)\, e^{2lz}\, dz$ would appear on the right-hand side of (6.1). Such a term does not appear because (6.1) refers to the rate of change of cross-wave energy within the rectangular box $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant \pi/l$, $-\infty < z$, rather than the energy between the wave-makers. The latter will certainly change at a rate given by the rate of working of the wave-makers against the pressure on them, but if we wish to consider the rate of change of energy in the rectangular box then we must consider the flux of energy across the boundaries as well as the rate of working against the pressure, and these contributions to some extent cancel.

Indeed, if E denotes the energy within the box, then

$$E = \int_0^{2\pi} dx \int_0^{\pi/l} dy \int_{-\infty}^{\zeta} dz (z + \tfrac{1}{2}u^2), \tag{6.2}$$

whence, using the free surface conditions (2.4), (2.5),

$$\frac{dE}{dt} = \int_0^{\pi/l} dy \int_{-\infty}^{\zeta} dz\, [\phi_t \phi_x]_{x=0}^{x=2\pi} \tag{6.3}$$

$$= -2\int_0^{\pi/l} dy \int_{-\infty}^{\zeta} dz [\dot{\beta} F(z)\, \phi_t|_{x=0}], \tag{6.4}$$

by symmetry and using (2.8). Thus the growth of the cross-waves within the box is due to the rate of working of the wave-makers against a stress $-\phi_t$. Equation (6.4) may be written (to third order) as

$$\frac{dE}{dt} = -2\int_0^{\pi/l} dy \left\{ \dot{\beta} F(0)\, \zeta\phi_t|_{x=0} + \int_{-\infty}^{0} dz \dot{\beta} F(z)\, \phi_t|_{x=0} \right\}, \tag{6.5}$$

in which we take the values of ζ, ϕ_t appropriate to a free cross-wave. The first term gives a contribution $-(\pi/l^2)\, F(0)\, \dot{\beta} c\ddot{c}$ to dE/dt. The second term contributes (to third order) only from the second-order term in ϕ_t independent of y. This may be obtained from (3.2) as $-\frac{1}{2}(c\ddot{c}+\dot{c}^2)$, which is the unattenuated second-order pressure discovered by Miche (1944) and explained physically by Longuet-Higgins & Ursell (1948) as due to the raising and lowering of the centre of gravity in a standing wave. When used in (6.5) it gives a contribution

$$\pi/l \int_{-\infty}^{0} F(z)\, dz \dot{\beta}(\dot{c}^2 + c\ddot{c}),$$

so that (6.5) agrees with (6.1). We note that for half-frequency cross-waves the second-order terms in c have the same frequency as β, so that the time-average of the right-hand side of (6.1) is indeed non-zero if the phase between the wave-maker and cross-waves is right.

7. Discussion

The theory described in this paper seems to provide a convincing explanation of the phenomenon of cross-waves in a closed tank with a wave-maker at one end. The theory agrees with the experiments of Lin & Howard (1960) in the following details:

(i) The frequency of the cross-waves is half that of the wave-maker, as observed. (Cross-waves at other multiples of the half-frequency are probably possible, though they would be much more difficult to excite.)

(ii) The frequency-amplitude curves of the cross-waves should lie to the right of the curves for free standing waves by an amount proportional to the amplitude of the wave-maker. The experimental results bear this out qualitatively, and are consistent quantitatively if one assumes the calculated frequency of free oscillations to be in error by a small amount independent of the wave-maker amplitude.

(iii) The phase relation between the wave-maker and cross-waves is as expected.

Viewed as a non-linear interaction, the cross-wave may be regarded as two progressive waves, of the form $\cos(ly-\omega t)$ and $\cos(ly+\omega t)$, each interacting with the basic frequency 2ω to reinforce the other.

It should be emphasized that cross-waves are a transverse instability independent of the primary motion. Probably the best way to understand their generation is in terms of the work done by the wave-maker against transverse stresses associated with the cross-waves, as discussed in §6. It seems highly plausible that the same mechanism operates when the primary motion consists of progressive waves and the cross-waves are observed to decay away from the wave-maker. In (3.17) the amount of excitation possible is described by

$$\alpha/\pi \left| F(0) - 2l\int_{-\infty}^{0} F(z)\,dz \right|,$$

which may be written dimensionally as

$$\alpha/L \left| F(0) - \frac{2\pi}{W}\int_{-\infty}^{0} F(z^*)\,dz^* \right|.$$

Presumably for progressive primary waves we must replace L here by the scale length of the distance to which the cross-waves extend, but the other factors are unaltered. Thus cross-waves may be eliminated if we choose $F(z^*)$ such that

$$F(0) = \frac{2\pi}{W}\int_{-\infty}^{0} F(z^*)\,dz^*. \tag{7.1}$$

The distance to which cross-waves extend is unknown, but is probably determined by dissipative effects, in which case the resonance half-width will be decreased and cross-waves eliminated even if (7.1) is not quite satisfied. Of course cross-waves are only possible if the wave-maker frequency is sufficiently close to twice the natural frequency of a cross-wave with a half-integral number of wavelengths across the tank.

If L_p is the wavelength of the progressive waves being generated then (for deep water) W in (7.1) may be replaced by $2L_p$. If the wave-maker is of the flap type with depth H then (7.1) gives $1 = \pi H/2L_p$, so that cross-waves are eliminated if H is chosen equal to $2L_p/\pi$. For a plunger wave-maker with a vertical stroke the displacement is given by $G[z+\beta(t)]$, but if the stroke is small this is approximately $G'(z)\beta(t)$, and the above theory applies with $F(z) = G'(z)$. In particular, if the plunger is a triangular wedge of depth H, (7.1) implies the elimination of cross-waves if $H = L_p/\pi$.

This is somewhat speculative; further theoretical and experimental work is required for a full understanding of the cross-wave phenomenon when the wave tank does not have a rigid reflecting wall opposite the wave-maker.

Cross-waves of the form $\cos nly \cos(mx-\omega t)$ are probably not possible as the radiative damping would exceed the rate of excitation.

Another possible example of parametric resonance in a wave tank was observed by Bowen & Inman (1969). They found that surface waves incident on a shelving beach generated standing edge waves which were generally of the same frequency as the incident waves, though they did occasionally observe half-frequency edge waves which grew slowly to a very large amplitude. Edge waves have a frequency depending on the slope of the beach, so that half-frequency edge waves could perhaps be excited parametrically by harmonic variation of the slope of the free surface due to the incoming waves. Edge waves of the same frequency as the incident waves (which, unlike the half-frequency waves, were also observed in the field by Bowen & Inman) are presumably generated by some more comlicated mechanism, but as Bowen & Inman suggest the phenomenon may be related to that of cross-waves.

In conclusion, it is rather intriguing that a box of water with a flapping side should show the behaviour of resonant non-linear oscillators in one direction (the primary standing waves, see Taylor 1953) and parametrically resonant non-linear oscillators in the other direction.

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Appendix. Simple properties of Mathieu's equation

Mathieu's equation may be written

$$\ddot{u} + \omega_0^2(1+\gamma\cos\sigma t)\,u = 0, \tag{A 1}$$

which describes the motion of a simple harmonic oscillator in the situation where some external parameter defining its frequency varies sinusoidally in time. If γ is small it is well known (e.g. Bogoliubov & Mitropolsky 1961, §17) that for σ

within $O(\gamma^N\omega_0)$ of $2\omega_0/N$ solutions exist of frequency $\frac{1}{2}N\sigma$ growing like e^{st} where s is also $O(\gamma^N\omega_0)$.

Some standard results for $N = 1$ are summarized here for convenient reference. If $\sigma = 2\omega_0+\delta$, an approximate solution,

$$u = A(t)\cos(\omega+\tfrac{1}{2}\delta)\,t+B(t)\sin(\omega+\tfrac{1}{2}\delta)\,t, \tag{A 2}$$

has

$$2\dot{A}+(\delta+\tfrac{1}{2}\gamma\omega_0)\,B = 0, \tag{A 3}$$

$$2\dot{B}-(\delta-\tfrac{1}{2}\gamma\omega_0)\,A = 0, \tag{A 4}$$

so that A, B are proportional to e^{st} where

$$s^2 = \tfrac{1}{4}[(\tfrac{1}{2}\gamma\omega_0)^2-\delta^2], \tag{A 5}$$

and grow or decay exponentially if $-\delta_m < \delta < \delta_m$, where

$$\delta_m = \tfrac{1}{2}\gamma\omega_0. \tag{A 6}$$

Note that $B = 0$ when $\delta = \frac{1}{2}\gamma\omega_0$ and $A = 0$ when $\delta = -\frac{1}{2}\gamma\omega_0$.

If we include in (A 1) a small frictional term $2\mu\dot{u}$, which would damp free oscillations like $e^{-\mu t}$, then A, B grow or decay exponentially for $-\delta_m < \delta < \delta_m$, where

$$\delta_m = [(\tfrac{1}{2}\gamma\omega_0)^2-4\mu^2]^{\frac{1}{2}}. \tag{A 7}$$

The presence of dissipation also alters the phase relation at $\delta = \pm\delta_m$. Taking $\gamma < 0$ and $\delta = +\delta_m$ (to tie in with §5),

$$A = \frac{2\mu}{\delta_m-\frac{1}{2}\gamma\omega_0}\,B, \tag{A 8}$$

so that for μ small $A \doteqdot (\mu/\delta_m)\,B$ and the steady solution at $\delta = \delta_m$ is now proportional to $\sin(\frac{1}{2}\sigma t+\mu/\delta_m)$ rather than $\sin\frac{1}{2}\sigma t$.

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